

USEFUL MATHEMATICAL RELATIONSHIPS

1. Trigonometric relationships

Pythagorean relationships

$$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

Angle sum and angle difference relationships

$$\begin{aligned}\sin(a+b) &= \sin(a) \cos(b) + \cos(a) \sin(b) \\ \sin(a-b) &= \sin(a) \cos(b) - \cos(a) \sin(b) \\ \cos(a+b) &= \cos(a) \cos(b) - \sin(a) \sin(b) \\ \cos(a-b) &= \cos(a) \cos(b) + \sin(a) \sin(b)\end{aligned}$$

Function-product relationships

$$\begin{aligned}\sin(a) \sin(b) &= \frac{1}{2} \cos(a-b) - \frac{1}{2} \cos(a+b) \\ \cos(a) \cos(b) &= \frac{1}{2} \cos(a-b) + \frac{1}{2} \cos(a+b) \\ \sin(a) \cos(b) &= \frac{1}{2} \sin(a+b) + \frac{1}{2} \sin(a-b) \\ \cos(a) \sin(b) &= \frac{1}{2} \sin(a+b) - \frac{1}{2} \sin(a-b)\end{aligned}$$

Function-sum and function-difference relationships

$$\begin{aligned}\sin(a) + \sin(b) &= 2 \sin[(a+b)/2] \cos[(a-b)/2] \\ \sin(a) - \sin(b) &= 2 \cos[(a+b)/2] \sin[(a-b)/2] \\ \cos(a) + \cos(b) &= 2 \cos[(a+b)/2] \cos[(a-b)/2] \\ \cos(a) - \cos(b) &= -2 \sin[(a+b)/2] \sin[(a-b)/2]\end{aligned}$$

Euler relationships

$$\begin{aligned}e^{ix} &= \cos(x) + i \sin(x) & i &= (-1)^{1/2} \\ \sin(x) &= (e^{ix} - e^{-ix})/2i & \cos(x) &= (e^{ix} + e^{-ix})/2\end{aligned}$$

Power series expansions for sin and cosine

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots & \sin(kx) &= \frac{kx}{1!} - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + \dots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots & \cos(kx) &= 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + \dots\end{aligned}$$

2. Indefinite integrals

$$290. \int (\sin ax) dx = -\frac{1}{a} \cos ax$$

$$291. \int (\cos ax) dx = \frac{1}{a} \sin ax$$

$$296. \int (\sin^2 ax) dx = -\frac{1}{2a} \cos ax \sin ax + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4a} \sin 2ax$$

$$297. \int (\sin^3 ax) dx = -\frac{1}{3a} (\cos ax)(\sin^2 ax + 2)$$

$$298. \int (\sin^4 ax) dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

$$299. \int (\sin^n ax) dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int (\sin^{n-2} ax) dx$$

$$302. \int (\cos^2 ax) dx = \frac{1}{2a} \sin ax \cos ax + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4a} \sin 2ax$$

$$303. \int (\cos^3 ax) dx = \frac{1}{3a} (\sin ax)(\cos^2 ax + 2)$$

$$304. \int (\cos^4 ax) dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

$$305. \int (\cos^n ax) dx = \frac{1}{na} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \int (\cos^{n-2} ax) dx$$

$$316. \int (\sin mx)(\sin nx) dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

$$317. \int (\cos mx)(\cos nx) dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

$$318. \int (\sin ax)(\cos ax) dx = \frac{1}{2a} \sin^2 ax$$

$$319. \int (\sin mx)(\cos nx) dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

$$320. \int (\sin^2 ax)(\cos^2 ax) dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}$$

$$321. \int (\sin ax)(\cos^m ax) dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

$$322. \int (\sin^m ax)(\cos ax) dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

$$389. \int x(\sin ax) dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$390. \int x^2(\sin ax) dx = \frac{2x}{a^2} \sin ax - \frac{a^2 x^2 - 2}{a^3} \cos ax$$

$$391. \int x^3(\sin ax) dx = \frac{3a^2 x^2 - 6}{a^4} \sin ax - \frac{a^2 x^3 - 6x}{a^3} \cos ax$$

$$393. \int x(\cos ax) dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$394. \int x^2(\cos ax) dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax$$

$$395. \int x^3(\cos ax) dx = \frac{3a^2 x^2 - 6}{a^4} \cos ax + \frac{a^2 x^3 - 6x}{a^3} \sin ax$$

$$399. \int x(\sin^2 ax) dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

$$400. \int x^2(\sin^2 ax) dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax - \frac{x \cos 2ax}{4a^2}$$

$$401. \int x(\sin^3 ax) dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}$$

$$402. \int x(\cos^2 ax) dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

$$403. \int x^2(\cos^2 ax) dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax + \frac{x \cos 2ax}{4a^2}$$

$$404. \int x(\cos^3 ax) dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}$$

3. Definite integrals

$$661. \int_0^{\infty} x^n e^{-ax} dx = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ \text{or} \\ \frac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$$

$$662. \int_0^{\infty} x^n \exp(-ax^p) dx = \frac{\Gamma(k)}{pa^k}, \quad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p} \right)$$

$$663. \int_0^{\infty} e^{-a^2x^2} dx = \frac{1}{2a} \sqrt{\pi} = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right), \quad (a > 0)$$

$$664. \int_0^{\infty} x e^{-x^2} dx = \frac{1}{2}$$

$$665. \int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

$$666. \int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$667. \int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$$